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# $\mathcal{PT}$-symmetrically regularized Eckart, Pöschl–Teller and Hulthén potentials

Miloslav Znojil
Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic
E-mail: znojil@ujf.cas.cz

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**Abstract.** $\mathcal{PT}$ (= parity times time-reversal) symmetry of complex Hamiltonians with real spectra is usually interpreted as a weaker mathematical substitute for Hermiticity. Perhaps an equally important role is played by the related strengthened analyticity assumptions. In a constructive illustration we complexify a few potentials solvable only in $s$-wave. Then we continue their domain from the semi-axis to the whole axis and obtain new exactly solvable models. Their energies turn out to be real as expected. The new one-dimensional spectra themselves differ quite significantly from their $s$-wave predecessors.

## 1. Introduction

Our mathematical understanding of many physical systems can become drastically simplified after their suitable complexification. This is true, first of all, in the study of resonances [1] and of several other quantum scattering phenomena [2]. Recently, the idea of working in a complexified phase space for bound states [3] re-entered the scene with a new enthusiasm supported by an immediate relevance of the related break-down of parity $\mathcal{P}$ in certain field theories [4].

In the mathematically more accessible quantum mechanical models certain exceptional complex interactions with $\mathcal{PT}$ symmetry happen to become strictly equivalent to a real potential after a supersymmetric [5] or integral, Fourier-like [6] transformation. For other models, the analysis of the related purely real spectra of energies has been performed by several techniques. One may recollect, e.g., the most straightforward numerical experiments [7], semi-classical approximants [8] and the so-called delta expansions [9]. Resummations of divergent perturbation series [10] and the so-called exact WKB method [11] also offered several Hamiltonians for which the spectra of energies $E_n$ were proved to be strictly real.

One of the most immediate sources of information about the possible connection or correlation between the absence of a decay Im $E_n = 0$ and the $\mathcal{PT}$-symmetry $H = \mathcal{PT}H\mathcal{PT}$ itself is provided by the exactly solvable models in one dimension. Step by step, there were proposed the $\mathcal{PT}$-symmetric versions of the harmonic oscillator [8], of the asymmetric Morse interaction [12], of the asymptotically symmetric (sometimes called 'scarf') hyperbolic oscillator [13] and of its asymptotically asymmetric but locally not too dissimilar (also known as Rosen–Morse) alternative [14]. Before complexification, all of them belong among the so-called shape invariant potentials (cf the review [15]), so some of their properties can be clarified using the language of supersymmetry [16].

On the basis of numerical experience [9] current attention is exclusively paid to the forces $V(x)$ which are analytic in $x$. An extremely interesting byproduct of this point of view can be found in a transition to more dimensions for quartic (i.e. unsolvable) oscillators [6] and for the central and exactly solvable $\mathcal{PT}$-symmetrized harmonic oscillator [17] and Coulomb problem [18]. Within the set of similar forces with a centrifugal-like singularity there still exist a few models without a clear interpretation. After a glimpse of table 4.1 of the review [15] we immediately discover two of them, namely, the Eckart model

$$V^{(\mathrm{Eck})}(r) = \frac{A(A-1)}{\sinh^2 r} - 2B\frac{\cosh x}{\sinh r} \tag{1}$$

and the generalized Pöschl–Teller potential

$$V^{(\mathrm{PT})}(r) = -\frac{A(A+1)}{\cosh^2 r} + \frac{B(B-1)}{\sinh^2 r}. \tag{2}$$

In the standard interpretation [2], both these $s$-wave models are only partially, *incompletely* solvable and, in this sense, lie somewhere in a 'no-man's land'. This was the main source of our present inspiration. We see no reason why these two interactions should not be appropriately continued to the whole line and classified, afterwards, as the two new or 'forgotten' exactly solvable $\mathcal{PT}$-symmetric models. This will be done in detail in sections 2 and 3 below.

We have to remind the reader that the latter force (2) may be often found in the current literature in its alternative form $V^{(\mathrm{GPT})}(r) = (u + v\cosh 2r)/\sinh^2 2r$ [15] or in the special form known as the Hulthén potential [2]. The former correspondence is mediated by the trivial re-scaling of the axis of coordinates by a factor of two. In the present context, the latter, much less trivial relationship deserves more explicit attention. Its thorough discussion will be included here, therefore, in section 4. A few further relevant overall comments may be found in our summary and final discussion in section 5.

## 2. $\mathcal{PT}$-regularization and Eckart oscillator

From the purely historical point of view the loss of Hermiticity in the domain of complex couplings proved to be more than compensated by the new insight into the solutions of one of the most popular unsolvable models $V(x) = \omega x^2 + \lambda x^4$ [19]. Today, its spectrum is understood as a *single* multi-sheeted analytic function of the complex coupling constant $\lambda \in \mathbb{C}$. The same idea applies to the set of resonances in the cubic well $V(x) = \omega x^2 + \lambda x^3$ although a careful analytic continuation must be also performed in the coordinate $x$ itself [7]. These observations guided the semi-classical and numerical studies of the forces $V^{(\delta)}(x) = \omega x^2 + gx^2(\mathrm{i}x)^\delta$ containing a variable real exponent $\delta$ [9]. The related $\mathcal{PT}$-symmetric quantum mechanics with its new perturbation series [20] as well as quasi-classical approximation schemes [11] and matrix-truncation methods [21] works with the globally, asymptotically deformed paths of integration in the related Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right]\psi(x) = E\psi(x). \tag{3}$$

The $\delta = 2$, quartic anharmonic oscillator of [22] exemplifies these systems, which need not remain integrable on the real line. Its asymptotic integrability and decrease of wavefunctions is only recovered after we bend both our coordinate semi-axes downwards and replace

$$\{x \gg 1\} \longrightarrow \{x = \varrho \mathrm{e}^{-\mathrm{i}\varphi}\} \qquad \{x \ll -1\} \longrightarrow \{x = -\varrho \mathrm{e}^{\mathrm{i}\varphi}\}$$

beyond a certain distance $\varrho_0 \gg 1$ and within certain bounds upon $\varphi \in (0, \pi/3)$. The further growth of $\delta$ beyond $\delta = 2$ would make both the asymptotical $\varphi$-wedges shrink and rotate downwards in the complex plane.

Let us defer the discussion of the similar cases to our last section 4 below. Returning now just to our first two examples (1) and (2) we may notice that both of them may be characterized by a 'weak', $\varphi = 0$ option. Globally they do not leave the real axis of $x$ at all. Such a simplification proves most natural in the $\delta \to \infty$ regular model of [23], admitting the most natural physical interpretation of the real physical coordinates after all. The related $\mathcal{PT}$-symmetrized oscillators need not necessarily differ from their Hermitian counterparts too much. One can hope to encounter just slight modifications of the formulae available, e.g., in the factorization context [24] and in its Lie-algebraic [25], operator [26] or supersymmetric [27] re-interpretations.

Equally straightforward innovations may be expected in the domain of our singular forces (1) and (2). One can simply *avoid* their isolated singularities by a *local* deformation of the integration path. In this way the strong repulsion at the origin (so popular in some phenomenological models [28] *and* fully impenetrable in one dimension) becomes readily tractable via a suitable choice of the cut.

### *2.1. Terminating solutions revisited*

Once we pay attention to the real $s$-wave potential (1) with the strongly singular core, usually attributed to Eckart [29], we have to keep in mind that this Hermitian model is solvable on the half-line only, with $r \in (0, \infty)$ and, conventionally, $A > \frac{1}{2}$ and $B > A^2$. Its fixed value of the angular momentum $\ell = 0$ is in effect a non-locality, which lowers its practical relevance in three and more dimensions.

As already mentioned, the local deformation of the integration path enables us to forget about the strong singularity at the origin. We may admit the presence of the so-called irregular components in $\psi(r) \sim r^{1-A}$ near $r = 0$. They would be, of course, unphysical in the usual formalism [30]. Here, on the contrary, we continue $r \to x$ with $x \in (-\infty, \infty)$ and encounter new possibilities.

In the new perspective we have to re-analyse the whole Schrödinger equation anew. Our choice of appropriate variables

$$\psi(x) = (y-1)^u (y+1)^v \varphi\left(\frac{1-y}{2}\right) \qquad y = \frac{\cosh x}{\sinh x} = 1 - 2z$$

is dictated by the arguments of Lévai [27], and the consequent $\mathcal{PT}$-symmetry considerations require that we use the purely imaginary couplings $B = \mathrm{i}\beta$. Then we insert $V^{(\mathrm{Eck})}(x)$ in equation (3) and our change of variables leads to its new form

$$z(1-z)\varphi''(z) + [c - (a+b+1)z]\varphi'(z) - ab\varphi(z) = 0 \tag{4}$$

where

$$c = 1 + 2u \qquad a + b = 2u + 2v + 1 \qquad ab = (u+v)(u+v+1) + A(1-A) \tag{5}$$

and

$$4v^2 = 2B - E \qquad 4u^2 = -2B - E. \tag{6}$$

Our differential equation is of the Gauss hypergeometric type and its general solution is well known [31],

$$\varphi(z) = C_1 \cdot {}_2F_1(a, b; c; z) + C_2 \cdot z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z). \tag{7}$$

Besides the obvious relevance of such an exceptional solvability of a model with a strong singularity in quantum mechanics, an independent encouragement of its study is provided by its methodical appeal in the context of field theory, especially in connection with the so-called Klauder phenomenon [32].

*2.2. Asymptotic boundary conditions*

Technically, the first thing we notice is that our parameters $a$ and $b$ are merely functions of the sum $u + v$ and vice versa, $u + v = (a + b - 1)/2$. The immediate insertion then gives the rule $(a - b)^2 = (2A - 1)^2$ and we may eliminate

$$a = b \pm (2A - 1). \tag{8}$$

We assume that our solutions obey the standard oscillation theorems [33] and become compatible with the boundary conditions $\psi(\pm\infty) = 0$ in equation (3) at a discrete set of energies, i.e. if and only if the infinite series ${}_2F_1$ terminate. Due to the complete $a \leftrightarrow b$ symmetry, we only have to distinguish between the two possible choices of $C_2 = 0$ and $C_1 = 0$.

In the former case with the convenient $b = -N$ (= non-positive integer) the resulting numbers $a + b$ and $u + v$ both prove to be real. Using the definition of $B$ the difference $u - v = -\mathrm{i}\beta/(u + v)$ comes out purely imaginary. The related terminating wavefunction series (7), i.e.

$$\psi(x) = \left(\frac{1}{\sinh x}\right)^{u+v} \mathrm{e}^{(v-u)x} \cdot \varphi[z(x)] \tag{9}$$

is asymptotically normalizable if and only if $u + v > 0$. This condition fixes the sign in equation (8) and gives the explicit values of all the necessary parameters,

$$a = 2A - N - 1 \qquad u + v = A - N - 1 \qquad u - v = -\mathrm{i}\frac{\beta}{A - N - 1}. \tag{10}$$

For all the non-negative integers $N \leqslant N_{\max} < A - 1$ the spectrum of energies is obtained in the following closed form:

$$E = -\frac{1}{2}(u^2 + v^2) = -(A - N - 1)^2 + \frac{\beta^2}{(A - N - 1)^2} \qquad N = 0, 1, \ldots, N_{\max}. \tag{11}$$

The normalizable wavefunctions become proportional to Jacobi polynomials,

$$\varphi[z(x)] = \text{const} \cdot P_N^{(u/2, v/2)}(\coth x). \tag{12}$$

We have shortly to return to the second option with $C_1 = 0$ in equation (7). Curiously enough, this does not bring us anything new. Although the second Gauss series terminates at a different $b = c - 1 - N$, the factor $z^{1-c}$ changes the asymptotics and one only reproduces the former solution. All the differences prove purely formal. In the language of our formulae one just replaces $u$ by $-u$ in (and only in) *both* equations (9) and (10). No change occurs in polynomial (12).

## 3. Pöschl–Teller potential

The Schrödinger equation (3) with the bell-shaped potential $V(r) \sim 1/\cosh^2 r$ is one of the most popular exactly solvable models in quantum mechanics. Its applications range from analyses of stability and quantization of solitons [34] to phenomenological studies in atomic and molecular physics [35], chemistry [36], biophysics [37] and astrophysics [38]. Its appeal involves solvability by different methods [27] as well as a remarkable role in scattering [2]. Its bound-state wavefunctions represented by Jacobi polynomials are also encountered as superpartners of a complex 'scarf' model [13].

Not too surprisingly, virtually all these applications lose their physical ground after addition of the repulsive spike. Still, it is not too difficult to extend the exact solvability itself to the latter potential, called, often, the Pöschl–Teller well [39]. The related Schrödinger equation (3) must be confined to semi-axis $r \in (0, \infty)$ or appropriately regularized.

### 3.1. Regularization

We may repeat that the impossibility of using the real $V^{(\mathrm{PT})}$ of equation (2) with $A > B > 0$ in more dimensions or on the whole axis in one dimension is felt to be unfortunate in methodical considerations and in perturbation theory [40]. Singularities of the centrifugal type are encountered in phenomenological models [28, 41] but, unfortunately, not many of them are solvable [42].

In our present regularization of the singularity we shall not deform the straight integration path at all. We shall rather proceed in a way inspired by the pioneering paper [6] where Buslaev and Grecchi employed simply a constant downward shift of the whole coordinate axis,

$$r = x - \mathrm{i}\varepsilon \qquad x \in (-\infty, \infty). \tag{13}$$

In a way similar to the oscillator $V^{(\mathrm{BB})}(x) = V^{(\mathrm{HO})}(x - \mathrm{i}c) = x^2 - 2\mathrm{i}cx - c^2$ of [8] and to its three-dimensional generalization [17] the meaning of the $\mathcal{PT}$-symmetry degenerates here to mere trivial invariance with respect to the simultaneous reflection $x \to -x$ and complex conjugation $\mathrm{i} \to -\mathrm{i}$. The shift (13) is the main source of *regularization* here. As long as $1/(x - \mathrm{i}\varepsilon)^2 = (x + \mathrm{i}\varepsilon)^2/(x^2 + \varepsilon^2)^2$ at any $\varepsilon \neq 0$, the centrifugal term remains nicely bounded in a way which is uniform with respect to $x$. Without any difficulties one is able to work with similar centrifugal-like terms on the whole real line of $x$.

The same idea applies to the regularized Pöschl–Teller potential

$$V^{(\mathrm{RPT})}(x) = V^{(\mathrm{PT})}(x - \mathrm{i}\varepsilon) \qquad 0 < \varepsilon < \pi/2.$$

This potential is a simple function of the Lévai's [27] variable $g(r) = \cosh 2r$. As long as $g(x - \mathrm{i}\varepsilon) = \cosh 2x \cos 2\varepsilon - \mathrm{i} \sinh 2x \sin 2\varepsilon$, the new force is $\mathcal{PT}$-symmetric on the real line of $x \in (-\infty, \infty)$,

$$V^{(\mathrm{RPT})}(-x) = [V^{(\mathrm{RPT})}(x)]^*.$$

Due to the estimates $|\sinh^2(x - \mathrm{i}\varepsilon)|^2 = \sinh^2 x \cos^2 \varepsilon + \cosh^2 x \sin^2 \varepsilon = \sinh^2 x + \sin^2 \varepsilon$ and $|\cosh^2(x - \mathrm{i}\varepsilon)|^2 = \sinh^2 x + \cos^2 \varepsilon$ the regularity of $V^{(\mathrm{RPT})}(x)$ is guaranteed for any parameter $\varepsilon \in (0, \pi/2)$.

### 3.2. Solutions

In a way parallelling the preceding section the mere analytic continuation of the $s$-wave bound states does not give the complete solution. One must return to the original differential equation (3). There we may conveniently fix $A + \frac{1}{2} = \alpha > 0$ and $B - \frac{1}{2} = \beta > 0$ and write

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\beta^2 - \frac{1}{4}}{\sinh^2 r(x)} - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 r(x)}\right)\psi(x) = E\psi(x) \qquad r(x) = x - \mathrm{i}\varepsilon. \tag{14}$$

This is the Gauss differential equation

$$z(1 + z)\varphi''(z) + [c + (a + b + 1)z]\varphi'(z) + ab\varphi(z) = 0 \tag{15}$$

in the new variables

$$\psi(x) = z^{\mu}(1 + z)^{\nu}\varphi(z) \qquad z = \sinh^2 r(x)$$

using the suitable re-parametrizations

$$\alpha^2 = (2\nu - \tfrac{1}{2})^2 \qquad \beta^2 = (2\mu - \tfrac{1}{2})^2$$

$$2\mu + \tfrac{1}{2} = c \qquad 2\mu + 2\nu = a + b \qquad E = -(a - b)^2.$$

In the new notation we have the wavefunctions

$$\psi(x) = \sinh^{\tau\beta+1/2}[r(x)] \cosh^{\sigma\alpha+1/2}[r(x)]\varphi[z(x)] \tag{16}$$

with the sign ambiguities $\tau = \pm 1$ and $\sigma = \pm 1$ in $2\mu = \tau\beta + \frac{1}{2}$ and $2\nu = \sigma\alpha + \frac{1}{2}$. This formula contains the general solution of hypergeometric equation (15),

$$\varphi(z) = C_1 {}_2F_1(a, b; c; -z) + C_2 z^{1-c} {}_2F_1(a + 1 - c, b + 1 - c; 2 - c; -z). \tag{17}$$

The solution obeys the complex version of the Sturm–Liouville oscillation theorem [33]. In the case of the discrete spectrum this means that we have to demand the termination of our infinite hypergeometric series, suppressing its undesirable asymptotic growth at $x \to \pm\infty$.

In a deeper analysis let us first put $C_2 = 0$. We may satisfy the termination condition by the non-positive integer choice of $b = -N$. This implies that $a = N + 1 + \sigma\alpha + \tau\beta$ is real and that our wavefunction may be made asymptotically (exponentially) vanishing under certain conditions. Inspection of formula (16) recovers that the boundary condition $\psi(\pm\infty) = 0$ will be satisfied if and only if

$$1 \leqslant 2N + 1 \leqslant 2N_{\max} + 1 < -\sigma\alpha - \tau\beta.$$

The closed Jacobi polynomial representation of the wavefunctions follows easily:

$$\varphi[z(x)] = C_1 \frac{N!\Gamma(1 + \tau\beta)}{\Gamma(N + 1 + \tau\beta)} P_N^{(\tau\beta,\sigma\alpha)}[\cosh 2r(x)].$$

The final insertions of parameters define the spectrum of energies,

$$E = -(2N + 1 + \sigma\alpha + \tau\beta)^2 < 0. \tag{18}$$

Now we have to return to equation (17) once more. A careful analysis of the other possibility $C_1 = 0$ does not recover anything new. The same solution is obtained, with $\tau$ replaced by $-\tau$. We may keep $C_2 = 0$ and mark the two independent solutions by the sign $\tau$. Once we define the maximal integers $N_{\max}^{(\sigma,\tau)}$ which are compatible with the inequality

$$2N_{\max}^{(\sigma,\tau)} + 1 < -\sigma\alpha - \tau\beta \tag{19}$$

we obtain the constraint $N \leqslant N_{\max}^{(\sigma,\tau)}$. The set of our main quantum numbers is finite.

## 4. Bent contours and Hulthén potentials

In both our above examples (1) and (2) an overall $\mathcal{PT}$-symmetry of the Hamiltonian is, presumably, responsible for the existence of the real and discrete spectrum [8]. Cannata *et al* [16] and Bender *et al* [43] were probably the first to notice that one of the various limits $\delta \to \infty$ of the power-law models with $\varphi \to \pi/2 - \mathcal{O}(1/\delta)$ becomes, unexpectedly, exactly solvable again, in terms of special Bessel functions. These observations attract attention to strongly deformed contours. One possibility for their interpretation is the Liouvillean change of variables [44].

### *4.1. The $\mathcal{PT}$-symmetry preserving changes of variables*

In the first step let us recollect that in the spirit of the old Liouville paper [45] the change of the (real) coordinates (say, $r \leftrightarrow \xi$) in the Schrödinger equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + W(r)\right]\chi(r) = -\kappa^2\chi(r) \tag{20}$$

mediates a transition to a different potential. In terms of an invertible function $r = r(\xi)$ which possesses a few first derivatives $r'(\xi), r''(\xi), \ldots$ we obtain the new bound state problem with the new interaction

$$V(\xi) - E = [r'(\xi)]^2\{W[r(\xi)] + \kappa^2\} + \frac{3}{4}\left[\frac{r''(\xi)}{r'(\xi)}\right]^2 - \frac{1}{2}\left[\frac{r'''(\xi)}{r'(\xi)}\right] \tag{21}$$

and normalizable wavefunctions

$$\Psi(\xi) = \frac{\chi[r(\xi)]}{\sqrt{r'(\xi)}}. \tag{22}$$

In the Jacobi-polynomial context the Liouvillean changes of variables have been applied systematically to all the Hermitian models (cf figure 5.1 in the review [15], or [46] for a more detailed illustration). A similar exhaustive study is still missing for the $\mathcal{PT}$-symmetric models within the same class. Let us now try to partially fill the gap. For the sake of brevity we shall restrict our attention to the $\mathcal{PT}$-symmetric initial equation (20) with the Pöschl–Teller potential

$$W(r) = \frac{\beta^2 - \frac{1}{4}}{\sinh^2 r} - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 r} \qquad r = x - \mathrm{i}\varepsilon \qquad x \in (-\infty, \infty). \tag{23}$$

Its normalizable bound states are proportional to the Jacobi polynomials,

$$\chi(r) = \sinh^{\tau\beta+1/2} r \cosh^{\sigma\alpha+1/2} r \, P_n^{(\tau\beta,\sigma\alpha)}(\cosh 2r)$$

at all the negative energies $-\kappa^2 < 0$ such that

$$\kappa = \kappa_n^{(\sigma,\tau)} = -\sigma\alpha - \tau\beta - 2n - 1 > 0.$$

These bound states are numbered by $n = 0, 1, \ldots, n_{\max}^{(\sigma,\tau)}$ and by the generalized parities $\sigma = \pm 1$ and $\tau = \pm 1$.

We may note that our initial $\mathcal{PT}$-symmetric model (20) remains manifestly regular provided only that its constant downward shift of the coordinates $r = r_{(x)} = x - \mathrm{i}\varepsilon$ remains constrained to a finite interval, $\varepsilon \in (0, \pi/2)$. In a key step of its present modification let us now change the coordinates as follows:

$$\sinh r_{(x)}(\xi) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\xi} \qquad \xi = v - \mathrm{i}u. \tag{24}$$

This shifts and removes the singularity at $r = 0$ to infinity ($u \to +\infty$). In an opposite direction, one cannot proceed equally easily from a choice of a realistic $V(\xi)$ to the re-constructed coordinate $r(\xi)$. This methodical asymmetry is due to the definition (21) containing the third derivatives and, hence, being too complicated to solve. Still we are quite lucky with our purely trial and error choice of equation (24). Firstly, the real line of $x$ becomes mapped onto a manifestly $\mathcal{PT}$-symmetric curve $\xi = v - \mathrm{i}u$ in accordance with the compact and invertible trigonometric rules

$$\sinh x \cos\varepsilon = \mathrm{e}^u \sin v$$
$$\cosh x \sin\varepsilon = \mathrm{e}^u \cos v$$

i.e. in such a way that

$$v = \arctan\left(\frac{\tanh x}{\tan\varepsilon}\right) = v_{(x)} \in (v_{(-\infty)}, v_{(\infty)}) \equiv \left(-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right)$$
$$u = u_{(x)} = \tfrac{1}{2}\ln(\sinh^2 x + \sin^2\varepsilon).$$

Our path of $\xi$ is a downward-bent arch which starts in its left imaginary minus infinity and ends in its right imaginary minus infinity while its top lies at $x = v = 0$ and $-u = -u_{(0)} = \ln 1/\sin\varepsilon > 0$. The top may move towards the singularity in a way mimicked by the diminishing shift $\varepsilon \to 0$. Although the singularity originally occurred at the finite value $r \to 0$, it has now been removed upwards, i.e. in the direction of $-u \to +\infty$.

### 4.2. Consequences

The first consequence of our particular change of variables (24) is that it does not change the asymptotics of the wavefunctions. As long as $r'(\xi) = \mathrm{i} \tanh r(\xi)$ the transition from equations (20) to (3) introduces just an inessential phase factor in $\Psi(\xi)$. This implies that the normalizability (at a physical energy) as well as its violations (at all other energies not belonging to the discrete spectrum) are both in a one-to-one correspondence.

The explicit relation between the old and new energies and couplings is not too complicated. Patient computations reveal its closed form. With a bit of luck, the solution proves to be non-numerical. The new form of the potential and of its binding energies is derived by the mere insertion in equation (21),

$$V(\xi) = \frac{A}{(1-\mathrm{e}^{2\mathrm{i}\xi})^2} + \frac{B}{1-\mathrm{e}^{2\mathrm{i}\xi}} \qquad E = \kappa^2. \tag{25}$$

At the imaginary $\xi$ and vanishing $A = 0$ this interaction coincides with the Hulthén potential.

In the new formula for the energies one has to notice their positivity. This is extremely interesting since the potential itself is asymptotically vanishing at both ends of its integration path. One may immediately recollect that a similar paradox has already been observed in a few other $\mathcal{PT}$-symmetric models with an asymptotic decrease of the potential to minus infinity [22, 47].

The exact solvability of our modified Hulthén potential is not yet guaranteed at all. A critical point is that the new couplings depend on the old energies and, hence, on the discrete quantum numbers $n$, $\sigma$ and $\tau$ in principle. This could induce an undesirable state dependence into our new potential. Vice versa, the closed solvability of the constraint which forbids this state dependence will be equivalent to the solvability at last. A removal of this obstacle means in effect a transfer of the state dependence (i.e. of the $n$-, $\sigma$- and $\tau$-dependence) in

$$A = A(\alpha) = 1 - \alpha^2 \qquad C(= A + B) = \kappa^2 - \beta^2$$

from $C$ to $\beta$. To this end, employing the known explicit form of $\kappa$ we may rewrite

$$C = C(\sigma, \tau, n) = (\sigma\alpha + 2n + 1)(\sigma\alpha + 2n + 1 + 2\tau\beta). \tag{26}$$

This formula is linear in $\tau\beta$ and, hence, its inversion is easy and defines the desirable state-dependent quantity $\beta = \beta(\sigma, \tau, n)$ as an elementary function of the constant $C$. The new energy spectrum acquires the closed form

$$E = E(\sigma, \tau, n) = A + B + \frac{1}{4}\left[\sigma\alpha + 2n + 1 - \frac{A + B}{\sigma\alpha + 2n + 1}\right]^2. \tag{27}$$

Our construction is complete. The range of the quantum numbers $n$, $\sigma$ and $\tau$ remains the same as above.

## 5. Discussion

### 5.1. Spectrum of the $\mathcal{PT}$-symmetric Eckart model

The new spectrum of energies seems phenomenologically appealing. The separate $N$th energy remains negative if and only if the imaginary coupling remains sufficiently weak, $\beta^2 < (A - N - 1)^4$. Vice versa, the highest energies may become positive, with $E = E(N_{\max})$ growing extremely quickly whenever the value of the coupling $A$ approaches its integer lower estimate $1 + N_{\max}$ from above. In this way, even a weak $\mathcal{PT}$-symmetric force $V^{(\mathrm{Eck})}(x)$ is able to produce a high-lying normalizable excitation. This feature does not seem to be connected to the presence of the singularity as it closely parallels the similar phenomenon observed for the $\mathcal{PT}$-symmetric Rosen–Morse oscillator which remains regular at the origin [14]. Also, in a way resembling harmonic oscillators the distance of levels in our model is safely bounded from below. Abbreviating $D = A - N - 1 = A_{\text{effective}} > 0$ its easy estimate

$$E_N - E_{N-1} = (2D + 1)\left(1 + \frac{\beta^2}{D^2(D+1)^2}\right) > 1$$

(useful, say, in perturbative considerations) may readily be improved to $E_N - E_{N-1} > \beta^2/D^2$ at small $D \ll 1$, to $E_N - E_{N-1} > 2D$ at large $D \gg 1$ and, in general, to an algebraic precise estimate obtainable, say, via MAPLE [48].

Let us emphasize in conclusion that the formulae we obtained are completely different from the usual Hermitian $s$-wave results as derived, say, by Lévai [27]. He had to start from the regularity at the origin which implied an opposite sign in equation (8). This had to end up with the constraint $B > 0$. Moreover, the size of $B$ limited the number of bound states.

In the present $\mathcal{PT}$-symmetric setting, a few paradoxes emerge in this comparison. Some of them may be directly related to the repulsive real core in our $V^{(\text{Eck})}(x)$ with imaginary $B$. Thus, one may notice that the *increase* of the real repulsion *lowers* the $N$th energy. In connection with that, the number of levels *grows* with the increase of coupling $A$. In effect, the new bound-state levels emerge as decreasing from the positive infinity(!). At the same time, the presence of the imaginary $B = \mathrm{i}\beta$ shifts the whole spectrum upwards in precisely the manner known from non-singular models.

### 5.2. *Paradoxes in the Pöschl–Teller case*

Let us now compare our final result (18) with the known $\varepsilon = 0$ formulae for $s$-waves [27]. An additional physical boundary condition must be imposed in the latter singular limit. This condition fixes the unique pair $\sigma = -1$ and $\tau = +1$. Thus, the set of $s$-wave energy levels $E_N$ is not empty if and only if $\alpha - \beta > 1$. In contrast, all our $\varepsilon > 0$ potentials acquire a uniform bound $|V^{(\text{RPT})}(x)| < \text{const} < \infty$. Due to their regularity, no additional constraint is needed. Our new spectrum $E_N^{(\sigma,\tau)}$ becomes richer. For sufficiently strong couplings it proves to be composed of three separate parts,

$$\begin{array}{lll} E_N^{(-,-)} < 0 & 0 \leqslant N \leqslant N_{\max}^{(-,-)} & \alpha + \beta > 1 \\ E_N^{(-,+)} < 0 & 0 \leqslant N \leqslant N_{\max}^{(-,+)} & \alpha > \beta + 1 \\ E_N^{(+,-)} < 0 & 0 \leqslant N \leqslant N_{\max}^{(+,-)} & \beta > \alpha + 1. \end{array} \tag{28}$$

The first one is non-empty at $A + B > 1$ (with our above separate conventions $A > -\frac{1}{2}$ and $B > \frac{1}{2}$). Concerning the latter two alternative sets, they may exist at $A > B$ or at $B > A + 2$, respectively. We may summarize that in a parallel to the $\mathcal{PT}$-symmetrized harmonic oscillator of [17] we have the $N_{\max}^{(-,+)} + 1$ quasi-odd or 'perturbed', analytically continued $s$-wave states (with a nodal zero near the origin) complemented by certain additional solutions.

In the first failure of a complete analogy the number $N_{\max}^{(-,-)} + 1$ of our quasi-even states proves systematically higher than $N_{\max}^{(-,+)} + 1$, especially at the larger 'repulsion' $\beta \gg 1$. This is a particular paradox, strengthened by the existence of another quasi-odd family which behaves very non-perturbatively. Its members (with the ground state $\psi_0^{(+,-)}(x) = \cosh^{A+1}[r(x)] \sinh^{1-B}[r(x)]$ etc) do not seem to have any $s$-wave analogue. They are formed at the prevalent repulsion $B > A+2$ which is even more counter-intuitive. The exact solvability of our example enables us to understand this apparent paradox clearly. In a way characteristic of many $\mathcal{PT}$-symmetric systems some of the states are bound by an antisymmetric imaginary well. A successful description of its perturbative forms $V(x) = \omega x^2 + \mathrm{i}\lambda x^3$ [7, 10] carries numerous analogies with the real and symmetric $V(x) = \omega x^2 + \lambda x^4$. A similar mechanism creates the states with $(\sigma, \tau) = (+, -)$ in the present example.

A significant novelty of our new model $V^{(\mathrm{RPT})}(x)$ lies in the dominance of its imaginary component *at short distances*, $x \approx 0$. Indeed, we may expand our force to first order in the small $\varepsilon > 0$. This gives the approximation

$$\frac{1}{\sinh^2(x - \mathrm{i}\varepsilon)} = \frac{\sinh^2(x + \mathrm{i}\varepsilon)}{(\sinh^2 x + \sin^2 \varepsilon)^2} = \frac{1}{\sinh^2 x} + 2\mathrm{i}\varepsilon \frac{\cosh x}{\sinh^3 x} + \mathcal{O}(\varepsilon^2). \tag{29}$$

We see immediately the clear prevalence of the imaginary part at short distances, especially at all the negligible $A = \mathcal{O}(\varepsilon^2)$.

An alternative approach to the above paradox may be mediated by a sudden transition from the domain of a small $\varepsilon \approx 0$ to the opposite extreme with $\varepsilon \approx \pi/2$. This is a shift which changes $\cosh x$ into $\sinh x$ and vice versa. It intertwines the role of $\alpha$ and $\beta$ as a strength of the smooth attraction and of the singular repulsion, respectively. The perturbative/non-perturbative interpretation of both our quasi-odd subsets of states becomes mutually interchanged near both the extremes of the parameter $\varepsilon$.

The dominant part (29) of our present model leaves its asymptotics comparatively irrelevant. In contrast to many other $\mathcal{PT}$-symmetric models as available in the current literature our potential vanishes asymptotically,

$$V^{(\mathrm{RPT})}(x) \to 0 \qquad x \to \pm\infty.$$

An introduction and analysis of continuous spectra in the $\mathcal{PT}$-symmetric quantum mechanics seems to be rendered possible at positive energies. This question will be left open here.

In the same spirit we may also touch the problem of the possible breakdown of the $\mathcal{PT}$-symmetry. In our present solvable example the violation of the $\mathcal{PT}$-symmetry is easily mimicked by the complex choice of the couplings $\alpha$ and $\beta$. Due to our closed formulae the energies will still stay real, provided only that $\mathrm{Im}\,(\sigma\alpha + \tau\beta) = 0$.

### *5.3. Transition to the Hulthén model*

In the light of our new results we may now split the whole family of the exactly solvable $\mathcal{PT}$-symmetric models which contain a strong singularity in the two distinct categories. The first one 'lives' on the real line and may be represented or illustrated not only by the popular Laguerre-solvable harmonic oscillator [17] but also by both our present Jacobi-solvable forces. The second category requires a arch-shaped path of integration which lies confined within a narrow vertical strip. It also involves both the Laguerre and Jacobi solvable subsets. The former may be represented by the complex Morse model of [12] and by the Coulomb force with a complex charge [18]. Our present new Hulthén example offers their first Jacobi-solvable counterpart. The parallels may be illustrated by the following picture:

$$\begin{array}{ccc} \boxed{\begin{array}{c}\text{symmetric}\\ V^{(HO)}(r)\ [17]\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{symmetric}\\ V^{(PT)}(r)\ [49]\end{array}} \\[2ex] \updownarrow\ r = -\mathrm{i}\exp\mathrm{i}\,x & & \updownarrow\ \sinh r = -\mathrm{i}\exp\mathrm{i}\,x \\[2ex] \boxed{\begin{array}{c}\text{periodic}\\ V^{(M)}(x)\ [12]\end{array}} & \longleftrightarrow & \boxed{\begin{array}{c}\text{periodic}\\ V^{(H)}(x)\ [50]\end{array}} \end{array}$$

where the vertical correspondence originates from the changes of variables. One notices the similarities in the (symmetric or periodic) form of the functions $V$ as well as the differences in the straight-line or bent-curve shapes of the domains $r = r(t) \in \mathbb{C}$ or $x = x(t) \in \mathbb{C}$, respectively.

The less formal difference between the two categories may be also sought in their immediate physical relevance. Applications of the former class may be facilitated by a limiting transition which is able to return them on the usual real line. In contrast, the second category may rather find its most useful place in the methodical considerations concerning, e.g., field theories and the mechanisms of parity breaking [4]. Within quantum mechanics itself the second category might also parallel the studies of the 'smoothed' square wells in a non-Hermitian setting [16, 43].

In conclusion, let us recollect that the $\mathcal{PT}$-symmetry of a Hamiltonian replaces and, in a way, generalizes its usual Hermiticity. This is the main reason why there exists an unexplored space for new solvable models. In their context, an example with an 'intermediate', hyperbola-shaped arc of coordinates remains still to be discovered. Up to now this type of contour has only been encountered in the 'quasi-solvable' (i.e. partially numerical) model of [22].

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